Threading Rotational Dynamics, Crystal Optics, and Quantum Mechanics to Risky Asset Selection

A

"Physics & Pizza on Wednesdays"

presentation

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Outline

Abstract

Physics is a discipline characterized by unity and universality. This presentation will attempt to illustrate how a basic concept finds application in diverse physical systems, and in the social sciences as well.

Introduction

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- In many cases, the material property is not rotationally symmetric, but varies from one direction to another: Think of a football (a prolate spheroid), and how its rotational inertia must be different in the oblong direction versus the direction normal to that.
- In such cases, and with a vector "rate quantity," the linear dependence must be extended to allow for each component of the dependent variable to depend on all three components of the independent quantity. For example, ...

The angular momentum of a rigid body of arbitrary shape is

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The manner of taming of this black beast is one of the most profound and versatile ideas of physics and mathematics.

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Yes, if you choose your coordinate axes judiciously! You can always find three distinguished directions, X, Y, Z, for the matrix/tensor of inertia, called its *characteristic directions*, or just *eigenvectors*, and in this case, *principal axes*, such that the desired simplification is realized:

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How do you find these magic directions? By solving eigenvalue equations, as in

$$\mathbb{I}\mathbf{n} = a\mathbf{n}, \tag{3}$$

which will in general give you three solutions for the magic directions **n**. You then choose the coordinate axes along these directions.

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Many crystals such as quartz have a symmetry axis, which is then necessarily a principal axis, the other two being any two directions orthogonal to that symmetry axis. There are then only two different indices of refraction, corresponding to the "fast" and "slow" directions, which give rise to birefringence.

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- His approach was based on the assumption that short-time changes in risky asset prices are best described as random fluctuations. Markowitz treated risky asset prices as normally distributed, correlated random variables, and characterized each asset price by a mean value and a variance, and the mutual interactions by a covariance, hence the designation "mean-variance" formulation.

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- The addition of a riskless asset (U.S. Treasury Bills, Notes, and Bonds) to the picture by Tobin a few years later completed the formulation of the basic model and produced a simple and intuitively appealing solution to the portfolio selection problem.

†M. H. Partovi and M. R. Caputo, *Principal Portfolios: Recasting the Efficient Frontier*, Economic Bulletin **7**, 1 (2004).

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Here's the efficient portfolio problem within this model:

$$\min_{\mathbf{x}} \mathbf{x}^{\dagger}\sigma\mathbf{x} \;\; s.t. \; 1 = \sum_{i=1}^{N} x_i \text{ and } \mathcal{R} = \mathbf{r}\cdot\mathbf{x}, \tag{4}$$

where $-\infty < x_i < +\infty$ represents the fraction of the total investment allocated to asset s_i, σ stands for the covariance matrix of the asset set, r_i represents the expected rate of return for asset s_i, $\mathcal{R}$ is the prescribed expected rate of return for the portfolio, $\mathbf{r}\cdot\mathbf{x} \stackrel{\text{def}}{=} \sum_{i=1}^{N} r_i x_i$, and "†" signifies matrix transposition.

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Note also that there are some $N(N+1)/2$ terms in this double sum, or more than 125000 entries for the $S\&P$ 500 asset set, involving 500 estimated rates of return and more than $125k$ estimated covariance factors!

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- If that is not bad enough, one essentially has to invert the resulting $N \times N$ covariance matrix to find the "efficient frontier."
- In response to these difficulties, a simplified model of stock price movements, namely the single-index model, was subsequently proposed by Sharpe (1963) and others, based on the idea that perfect equilibrium prevails in the capital markets. The single index of the model is the result of that equilibrium.
- Much of the current jargon used by "financial advisers" and such is actually rooted in the single index CAPM model.

Here's a recap of the single index model. Consider a set of N assets $\{s_i\}$, $1 \leq i \leq N$, whose rates of return are normally distributed random variables given by

$$\rho_i \stackrel{\text{def}}{=} \alpha_i + \beta_i \rho_{mkt}, \tag{6}$$

where α_i and ρ_{mkt} are uncorrelated, normally distributed random variables with expected values and variances equal to $\bar{\alpha}_i$, $\bar{\rho}_{mkt}$ and $\bar{\alpha_i^2}$, $\bar{\rho^2}_{mkt}$, respectively.

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The quantity β_i associated with asset s_i is a constant which measures the degree to which s_i is coupled to the overall market variations. Thus the attributes of a given asset are assumed to consist of a *market-driven* (or *systematic*) part described by $(\beta_i \rho_{mkt}, \beta_i^2 \bar{\rho^2}_{mkt})$ and a *residual* (or *specific*) part described by $(\alpha_i, \bar{\alpha_i^2})$, with the two parts being uncorrelated.

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Note the huge simplification in the covariance structure of this model as compared to the mean-variance model. The efficient frontier calculation is still a nontrivial, nonintuitive analysis.

The Principal Portfolio Environment

As in the physical examples of dynamics and optics, the covariance matrix σ admits a set of N orthogonal eigenvectors $\mathbf{e}^\mu$, $\mu = 1, 2, \ldots, N$, so that $\sigma \mathbf{e}^\mu = v_\mu^2 \mathbf{e}^\mu$, where $v_\mu^2 \geq 0$ are the eigenvalues of the covariance matrix.

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The principal portfolios are now defined by $S_\mu \stackrel{\text{def}}{=} \sum_{i=1}^N e_i^\mu s_i / W_\mu$, i.e., the principal portfolio S_μ is defined to be a mix which contains an amount e_i^μ / W_μ of asset s_i, where $W_\mu \stackrel{\text{def}}{=} \sum_{i=1}^N e_i^\mu$.

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Note that W_μ represents the relative investment weight of the μth eigenvector, a quantity which is in general different from unity and may even be negative if asset (short) sales dominate asset purchases in the mix that constitutes the portfolio.

The Principal Portfolio Theorem

Theorem

Every investment environment $\{s_i, r_i, \sigma_{ij}\}_{i,j=1}^{N}$ *which allows short sales can be recast as a principal portfolio environment* $\{S_\mu, R_\mu, \mathsf{V}_{\mu\nu}\}_{\mu,\nu=1}^{N}$ *where the principal* ***covariance matrix*** V ***is diagonal****. The weighted mean of the principal variances equals the mean variance of the original environment. In general, a typical principal portfolio is hedged and leveraged.*

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Observe that each principal portfolio is characterized by an expected rate of return $R_\mu = \sum_{i=1}^{N} e_i^\mu r_i / W_\mu$ and a variance ${V_\mu}^2 = \sum_{j=1}^{N} \sum_{i=1}^{N} e_i^\mu \sigma_{ij} e_j^\mu / {W_\mu}^2 = v_\mu^2 / {W_\mu}^2$, to be referred to as *expected principal rate of return* and *principal variance*, respectively. Note also that the expected principal rate of return is unrestricted in magnitude and sign, so that a negative R_μ is entirely possible.

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Would one want to deal with a principal portfolio with a **negative** expected return?

Asset Selection in the Principal Portfolio Environment

The allocation rule for an efficient portfolio with unlimited short selling and a riskless asset is

$$X_\mu(R_0, \mathbf{R}, \mathbf{Z}, \mathcal{R}) = A(R_\mu - R_0){Z_\mu}^2,\ 1 \leq \mu \leq N, \tag{7}$$

and

$$X_0(R_0, \mathbf{R}, \mathbf{Z}, \mathcal{R}) = 1 - \sum\nolimits_{\mu=1}^{N} X_\mu(R_0, \mathbf{R}, \mathbf{Z}, \mathcal{R}), \tag{8}$$

where A is given by

$$A \stackrel{\text{def}}{=} (\mathcal{R} - R_0)/[\sum\nolimits_{\mu=1}^{N}(R_\mu - R_0)^2 {Z_\mu}^2].$$

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$$X_0(R_0, \mathbf{R}, \mathbf{Z}, \mathcal{R}) = 1 - \sum\nolimits_{\mu=1}^{N} X_\mu(R_0, \mathbf{R}, \mathbf{Z}, \mathcal{R}), \tag{8}$$

where A is given by

$$A \stackrel{\text{def}}{=} (\mathcal{R} - R_0)/[\sum\nolimits_{\mu=1}^{N}(R_\mu - R_0)^2 {Z_\mu}^2].$$

The efficient portfolio return $\mathcal{R}$ is normally higher than the riskless return R_0, in which case A would be positive and proportional to the *return surplus* $\mathcal{R} - R_0$. Therefore, each principal portfolio is bought, or sold short, according to whether it produces a return surplus, or deficit, relative to the riskless asset, and in proportion to the said surplus or deficit and in inverse proportion to the portfolio variance.

Concluding Remarks

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- The fact that every symmetric (or Hermitian, if comples) matrix can be diagonalized and the original basis replaced with the principal basis is enormously useful and constitutes the foundation of many related concepts and methods in the physical sciences, engineering, and social sciences.