Quantitative Reasoning with Advanced Mathematical Topics

Preface

The **Quantitative Reasoning with Advanced Mathematical Topics (QRAT)** course was developed to better prepare college and career-bound high school seniors with the 21st Century skills necessary to meet the mathematical thinking and problem-solving expectations of higher education courses and workplace requirements. Through a collective impact model, Region 3 approaches student learning through intersegmental partnership agreements that include Sacramento State University, the local community colleges, county offices of education, and feeder high school districts. The success of the ESM stems from the fact that it is not only a living curriculum and pedagogy that is designed to meet the immediate needs of high school seniors, but it also embodies the structural flexibility to be informed by the vibrant intersegmental professional learning communities. Essentially, the partnership structure affords each educational segment the opportunity to collaboratively define the challenges around preparation in mathematics while providing the foundation to forge better-aligned instructional practices across schools, colleges, and universities for the success of our students.

Acknowledgements

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Introduction

Course Purpose and Goals

This course is designed to strengthen students' mathematical foundation and prepare students to be college and career ready. The goal of the course is to deepen conceptual understandings of mathematical theory, skills and strategies. The course is designed to incorporate National Common Core Standards for Mathematical Practice and is aligned with specific high school content standards listed in the California Common Core State Standards for Mathematics (CCCSS-M). Utilizing real-world applications, this course serves both college and career bound high school seniors.

The purpose of this course is to fulfill the need to provide more math options for high school seniors to take a mathematics course that better prepares them for transition into college-level work or workplace expectations. The target student cohort is seniors who are interested in a major that heavily depends on science, technology, engineering, and/or mathematics (STEM) and other math intensive fields. This course is designed to better prepare students for the rigorous mathematics requirements in those fields. It is also intended for high school seniors who have completed Algebra II or Integrated Math III and who have met the CAASPP/EAP standards but do not wish to take a precalculus or calculus course in high school.

The development of this course was predicated on the idea that students who had previously considered themselves as unsuccessful in mathematics could learn and thrive in an environment which fosters engagement and conceptual learning. With a focus on depth, not breadth, students would master mathematical content and be able to transfer their skills to college and to career pathways. Lessons and tasks provide students with opportunities to solve challenging problems in which they gather, analyze, and evaluate information, work effectively in groups to make decisions using critical reasoning skills, as well as opportunities to communicate concisely through written and oral language. Throughout the course, students increase their perseverance to make sense of and to solve real-world and theoretical mathematical problems, develop a greater perspective of underlying structures of mathematics and how they connect mathematical topics. Students gain an appreciation of mathematics and its applications and develop a growth mindset towards mathematics that enables the student to continue to persevere through problem solving in the quantitative reasoning nature of college-level courses.

Major Student Outcomes

- Develop a growth mindset towards mathematics that enables the student to continue to persevere through problem solving in higher level math courses.
- Become better problem solvers.
- Build critical thinking skills.
- Increase their perseverance to make sense of and to solve real-world and mathematical problems.
- Deepen their understanding of underlying structures of mathematics.
- Gain appreciation of mathematics and its applications.
- Improve their ability to communicate their mathematical thinking.
- Develop their ability to work effectively as a member of a team.

Course Objectives

- Demonstrate the Standards for Mathematical Practice when engaged in mathematics.
- Flexibly apply problem solving strategies (e.g., guess and check, logic/deductive reasoning, tables and lists...) to contextual situations to deepen conceptual understandings of the structures and applications of mathematics.
- Analyze the information imbedded in different types of contextual problems and determine what data is given and what assumptions can be justified.
- Identify and assess the importance of ambiguities and complexities within a problem.
- Strengthen number sense and procedural fluency.
- Make connections between numeric and algebraic expressions and representations.
- Examine and apply families of functions including: linear, quadratic, exponential, logarithmic, absolute value, and piece-wise.
- Make connections between conceptual categories of mathematical content.
- Utilize understanding of linear and exponential functions in financial contexts.
- Reflect on their work and edit for clarity and accuracy.
- Make generalizations based on observations and repeated reasoning.
- Communicate reasoning verbally and in writing.
- Provide an appropriate level of justification in an organized viable argument, free from logical and arithmetical errors.
- Work collaboratively within small groups.

Course Outline/Sequence of Learning

UNIT 1 Developing the Mathematical Practices

This unit establishes a foundation for the rest of the course, stressing skills such as quantitative reasoning, the analysis and use of structure and repeated reasoning, and perseverance in solving traditional as well as nontraditional problems. Students deepen their understanding and application of a variety of problem solving strategies. Each problem requires that students engage in multiple standards of mathematical practice as outlined in the California Common Core State Standards for Mathematics (CCCSS-M). This unit also provides opportunities and support for students to realize that they can learn mathematics by conjecturing, investigating and justifying. In addition, students will develop team-building skills, learn to work effectively in groups and enhance their verbal and written mathematical communication skills. Problem solving skills will be used throughout the course.

To extend student content knowledge into a new area of mathematics, this unit explores topics in graph theory and number theory. These tasks require students to take a contextual situation and represent it as a mathematical diagram that can be analyzed to identify specific conditions using information about edges and vertices. Students explore Euler circuits and paths and the role that edges and vertices play in them. Students informally derive Faulhaber's Formula for degrees one, two, and three.

UNIT 2 Algebraic Foundations for Higher Mathematics

The foundational approach of the course, in which students recognize and extend numeric patterns to generalize algebraic concepts, is established in this unit by an extension of linear and quadratic functions. The differences between discrete and continuous domains are explored through the transition between patterning and algebraic representations. Extending and deepening their knowledge from previous concepts to topics in number theory and calculus, students generate quadratic and linear finite difference tables which emphasize the hidden structure in polynomial sequences, laying the foundation for derivatives of polynomials and the Fundamental Theorem of Calculus. In addition, students explore the sum of arithmetic sequences using Gauss' method to derive the explicit formula for sequences with linear differences, paving the way for future exploration of polynomials and exponential functions. Students also make connections between composition of functions and parameterizations.

UNIT 3 Complex Numbers, Polar Coordinates and Vectors

This unit introduces complex numbers and operations involving complex numbers using the context of polynomial functions with imaginary solutions. Students find the conjugate of a complex number and discover its connection to moduli and quotients.

Students study connections between arithmetic operations and the graphical representation of complex numbers. They also develop an understanding of the connection between complex numbers represented by rectangular coordinates, polar coordinates, and vectors.

Note: This unit is a stand-alone unit and is movable within the remaining units. It is placed here because complex numbers are connected to roots of quadratic functions. Alternatively this unit may be addressed after Unit 7, Unit 8, or Unit 9.

UNIT 4 Polynomial and Rational Functions

Students deepen their understanding of polynomial functions and their graphs. They recognize and compare key features of these including symmetries, end behaviors, intercepts, and extrema (both global and local). Students explore and utilize the concept of Rolle's Theorem when identifying local behaviors. They extend these concepts to apply them to rational functions in general, adding the consideration of vertical and horizontal asymptotes, limits, and the behavior of graphs as they approach asymptotes. Students build functions by combining linear functions using all four operations. They use polynomial and other rational functions to model real world scenarios.

UNIT 5 Introduction to the Concepts of Calculus

Students will grapple with problems that develop a basic understanding of calculus. Building on and extending their use of difference tables developed in units 2 and 3, students begin to explore the relationship between the degrees of a polynomial and its rate of change, leading to the concept of a derivative and the derivative power rule. Students will formalize their understanding of Rolle's Theorem as it applies to polynomials, in contrast with its inapplicability to rational functions. Students formalize the sums of consecutive squares and the sums of consecutive cubes and connect the graphical representation of these sums with estimates of areas under a curve via Riemann sums. This development is extended to various discrete integration methods such as the trapezoidal, midpoint and right-hand or left-hand endpoint rules, ultimately exposing students conceptually to the Fundamental Theorem of Calculus.

UNIT 6 Inverse Functions

Students are given opportunities to develop conceptual understanding of inverse functions as well as fluency with writing equations, graphing, and solving while solidifying their understanding of exponential and logarithmic functions. They begin by exploring relationships between the properties of exponents and logarithms. The concept of inverses is then expanded to other invertible functions. Students express this relationship in terms of domain and range and to make connections to graphs of these functions. Use of inverse notation is introduced. Students will verify inverse relationships through composition of functions and the examination of graphs, identifying the line of reflection as $y = x$.

Throughout the unit students deepen their understanding of concepts through consideration of real-world contexts including biology, medical science, finance, and cryptography, building and interpreting models for given scenarios.

UNIT 7 Piecewise and Non-Invertible Functions

The unit provides opportunities for students to create, analyze, evaluate, and interpret advanced piecewise functions that model a variety of real-world and theoretical mathematical problems utilizing multiple representations (e.g. tables, charts, graphs). Students represent non-invertible functions as piecewise functions, in which each piece is invertible. The relationship between differentiability and continuity will be explored in reference to function types covered in previous units as well as piecewise functions and absolute value functions. In this unit, content development builds on algebraic and graphical representations of piecewise functions.

This unit begins by deepening student understanding of systems of linear equations by continuing the use of patterning developed throughout the course. Matrices are introduced as a new method to solving systems of linear equations. Patterning is then used to explore a more rigorous development of systems of inequalities. Students apply systems of inequalities to discuss feasible regions within a given context. Finally, students will apply these concepts to solve linear programming problems.

UNIT 8 Matrices and Linear Programming

Students deepen their understanding of systems linear equations through continued patterning of tables, graphs, and modeling techniques which have been developed throughout the course. Matrices are introduced and provide a new method for students to use in organizing information and solving systems of linear equations. Through patterning, students develop a rigorous understanding of systems of inequalities. Students apply systems of inequalities to investigate the concept of feasibility and determine feasible regions within given contexts. Students understand and apply objective functions allowing them to model and solve real-world problems using linear programming.

UNIT 9 Economic Applications

Through the use of modeling, interpretation, and analysis, students understand the connections between the mathematics they have learned and the financial literacy that is essential to a lifetime of fiscal well-being. Contexts to be explored include short-term and long-term savings, loans, annuities, and amortization. Tasks require students to write equations to model financial scenarios and products using linear, exponential, and quadratic functions as well as functions composed of multiple functions (e.g. to model amortization and annuitization). Students explore the structure of these functions to gain a deeper understanding of financial mathematics. They create amortization tables in addition to comparing various types of functions in the context of investment growth and loan costs. Students apply their knowledge of the derivative as the slope of the tangent line and interpret them in terms of different financial contexts. They also address the relationship between continuity and differentiability.

Pedagogy

The primary role of the teacher in this course is that of facilitator. There is an emphasis on facilitation of effective and purposeful team work that includes orchestrating students' academic discourse. The design of this course minimizes direct instruction and focuses rather on more interactive and collaborative strategies. Learning experiences provide opportunities for students to engage in problem solving, simulation, exploration, and discovery, as well as application and communication of mathematical thinking. Although there is an emphasis on group work, students are also expected to think independently about the content and to reflect on their own learning. In general, the instructional strategies incorporated should support a culture of growth mindset among a community of curious and persistent learners.

Team Building

Teamwork has become an important part of the working culture and many businesses now look at teamwork skills when evaluating a person for employment. Most companies realize that teamwork is important because either the product is sufficiently complex that it requires a team with multiple skills to produce and/or a better product will result when a team approach is taken. Therefore, it is important that students learn to function in a team environment so that they will have teamwork skills when they enter the workforce. Also, research tells us that students learn best from tasks that involve social interactions.

Collaborative learning should be included in almost every classroom, but some teachers struggle with having students work cooperatively. There are a number of reasons for this struggle, which include the need to develop good team exercises and the added difficulty in assessing the individual performance of the team members. This is where understanding how to teach effective teamwork becomes a crucial task for the teacher.

Characteristics of Teamwork

- Members work interdependently and work towards both personal and team goals, and they understand these goals are accomplished best by mutual support.
- Members feel a sense of ownership towards their role in the group because they committed themselves to goals they helped create.
- Members collaborate together and use their talent and experience to contribute to the success of the team's objectives.

- Members base their success on trust and encourage all members to express their opinions, varying views, and questions.
- Members make a conscious effort to be honest, respectful, and listen to every person's point of view.
- Members are encouraged to offer their skills and knowledge, and in turn each member is able to contribute to the group's success.
- Members see conflict as a part of human nature and they react to it by treating it as an opportunity to hear about new ideas and opinions. Everybody wants to resolve problems constructively.
- Members participate equally in decision-making, but each member understands that the leader might need to make the final decision if the team cannot come to a consensus agreement.

NDT Resource Center, Teaching Resources - Classroom Tips, Teamwork in the Classroom

Problem Solving Techniques

Two major goals of this course are that students become better problem solvers and that they increase their critical thinking skills. According to the National Council of Teachers of Mathematics, “Problem solving plays an important role in mathematics and should have a prominent role in the mathematics education of K-12 students.”

Retrieved from http://www.nctm.org/Research-and-Advocacy/research-brief-and-clips/Problem-Solving

Throughout this course students will be provided with regular opportunities to make sense of and solve problems that involve meaningful mathematics. Most of these problems may be approached in a variety of ways.

Problem solving techniques employed by students may be categorized as follows:

- Look for a Pattern
- Draw a Picture
- Guess and Check
- Work Backwards
- Use Small Numbers
- Break the Problem into Sub-Problems
- Make an Organized Table or List
- Write an Equation
- Use of Logic/Deductive Reasoning
- Act It Out

Assessment

Exit Slips

Exit slips are used as the daily formative assessment to help inform planning for future lessons. The intent is for teachers to monitor student learning and adapt lessons as necessary.

Journals

Journal entries are used as critical lesson-level formative assessments. Students are expected to re-engage with content and/or to rewrite their journal entries until mastery is evident.

Unit Assessments

Summative unit assessments may be comprised of a variety of components (e.g. projects, posters, presentations, pencil-paper exams). Unit assessments will involve both group and individual elements.